R.A. BASE

The Association
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Engineering and Shipbuilding
Draughtsmen.

STRESS DETERMINATION BY MEANS OF THE ELECTRICAL RESISTANCE-WIRE STRAIN GAUGE.

By Wm. ROBSON, B.Sc. (Eng.), London.

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STRESS DETERMINATION BY MEANS OF THE ELECTRICAL RESISTANCE-WIRE STRAIN GAUGE.

By Wm. Robson, B.Sc. (Eng.), London.

I.—INTRODUCTION.

The majority of machines and engineering structures may be said to fall into the category of highly redundant frames.

Under optimum load conditions—static or dynamic—experience shows that despite a generous factor of safety, failure sometimes occurs.

The determination of the position of the most highly stressed area is often a result of design intuition, or in the last resort, inspired guesswork. When approached by way of formal mathematics the solution of such a problem presents considerable difficulty. Where solutions do exist, the effect of fixing conditions often renders the result very much open to doubt.

In order to eliminate this element of uncertainty, as far as possible, the most recent trend is to support conclusions obtained from mathematical reasoning or *a priori* assumptions by direct evidence in the shape of results derived from experimental stress analysis.

Five practical methods of stress determination are in regular use, the respective "tools" being (1) photoelasticity; (2) brittle lacquers; (3) electrical resistance-wire strain gauges; (4) mechanical extensometers; and (5) acoustic strain gauges.

Tests are carried out either on prototypes or scale models. In the latter case multipliers are used for scale and mass effect.

For static tests, where the time of application of the load is under control, photoelasticity, brittle lacquers, and strain gauges of the mechanical, acoustic or electrical type may be used, since the strains vary slowly with time.

Under dynamic loading, the strains change rapidly, and instantaneous response to these changes must be an inherent property of the strain-measuring device.

By virtue of its simple construction, robustness, negligible inertia and linear response up to very high frequencies, the electrical resistance-wire strain gauge is the type usually selected.

In function, the electrical strain gauge simulates the action of the organic sensory nerve.

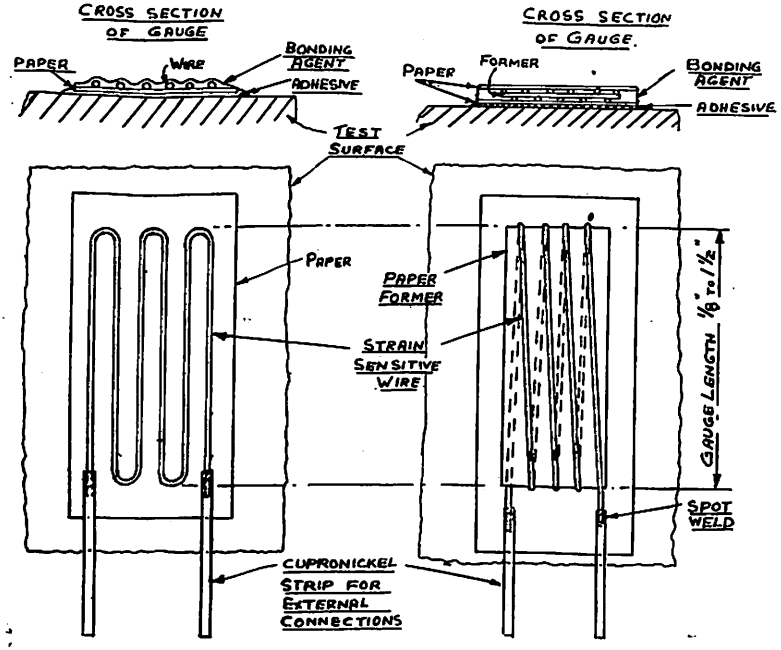
II.—CONSTRUCTION OF GAUGE.

In 1856 Lord Kelvin* discovered that the change in electrical resistance of a length of wire varied in direct proportion to the change in applied strain—an upper and lower limit of strain in the wire being understood.

During World War II the first resistance-wire strain gauge was evolved at the California Institute of Technology, U.S.A.

The strain-sensitive element consisted of a length of fine wire 1/1000 of an inch in diameter. This was formed into a flat grid or series of short parallel lengths joined by semi-circular arcs and cemented to the surface of a strip of thin tissue paper by means of a thermo-plastic cellulose adhesive. The construction of this American type strain gauge is shown in Fig. 1a.

In England the construction usually differs. The fine wire is closely wound on a resin-impregnated former. This eliminates the end loops. The result is a stiffer and less permeable gauge.



SINGLE LAYER TYPE — Fig. 1a.

WOUND TYPE — Fig. 1b.

Resistance Wire Strain Gauges.

*Phil. Trans. Royal Society, 146 (1856), p. 733.

In consequence, the drying-out of the solvent in the adhesive used to affix the gauge to the test surface is more protracted than in the single layer or flat grid type.

The double layer or wound type of strain-gauge is shown in Fig. 1b.

In a third type of gauge the fine wire, silk covered, forms the weft of a close-mesh fabric similar in appearance to artificial silk. Such a gauge is very thin and flexible, drys out quickly after fixing, and by reason of its limp, multicellular construction makes good adhesion with the test surface.

To facilitate connection to external leads, thin flat cupronickel strip is spot-welded to the fine strain-gauge wire. The portion of the strip external to the gauge solders easily and ensures good electrical connection.

In general, America favours the single layer, low-resistance strain-gauge (100-150 ohms), whereas in Britain the low-resistance type, and the high-resistance type (2000-10,000 ohms) are both in common use.

No particular merit is claimed for either type, but for dynamic measurements the high-resistance type strain-gauge is preferred, since it produces a larger signal that requires less amplification.

The gauge lengths of the various strain-gauges range from $1\frac{1}{2}$ inches down to $\frac{1}{8}$ inch. The steps are usually in the following order :—$1\frac{1}{2}''$, $1''$, $\frac{1}{2}''$, $\frac{1}{4}''$ and $\frac{1}{8}''$. For the wound type, the corresponding resistance values are usually 10,000, 2,500, 2,000, 200 and 120 ohms respectively. In a standard 2,500 ohm strain-gauge of 1 inch gauge length, the strain-sensitive wire has a length of about 40 inches and a diameter of ·001 inches. The material is an alloy of nickel and chromium known as nichrome.

When affixed to a test surface, a cross section would reveal a layer construction arranged in the following order :—Test surface—adhesive—paper—cellulose—wire—former—wire—cellulose—paper. Once used, the strain-gauge cannot be removed without being destroyed in the process. The maximum working temperature is about 60°C.

For temperatures above 60°C. the wire is embedded in a thermo-setting synthetic resin. The gauge is similar in construction to the cellulose type with the exception of the synthetic bonding agent. In order to affix the gauge, the phenol-resin back is held in contact with the test surface under light pressure and the gauge subjected to a steady temperature of 135°C. for two hours. This process polymerises or "cures" the phenol-resin and ensures good adhesion and insulating properties. Resin-backed gauges have a "shelf life" of approximately three months. After this period the adhesive property of the resin becomes uncertain. The highest temperature at which the gauge will function is limited by the

charring of the organic material used in the construction. A safe upper limit is 200°C.

For very high temperatures, wire bonded in situ, in a ceramic cement is used. Such gauges are obviously not commercially available.

III.—FUNDAMENTAL PRINCIPLES.

When a strain-gauge is affixed to a test surface by means of a suitable adhesive, another phenomenon displayed is that the *decrease* in resistance varies strictly in proportion to the change in *compressive* strain. This highly desirable characteristic is due to the fact that the high ratio of surface-area of wire to cross-sectional area of wire does not allow the wire freedom to "buckle up" in the matrix or bonding agent. The wire is in effect an integral part of the gauge.

If a graph is drawn with mechanical strain as abscissæ and

$$\text{Electrical Strain or } \frac{\text{Change in Gauge Resistance}}{\text{Initial Gauge Resistance}} = \frac{\delta R}{R}$$

as ordinate, it will be found that the resultant curve is a straight line through the origin. No transition "jog" takes place at this point. At a value of $\pm$ ·003 strain, corresponding to a direct stress of about $\pm$ 40 tons/in.^{2} in steel, the curve becomes non-linear. The part of the curve over which the proportional range extends is shown in Fig. 2.

Since the equation of a straight line through the origin is of the form $y = kx$, k being the slope, then from the graph, Fig. 2, we have

$$\frac{\text{Electrical Strain}}{\text{Mechanical Strain}} = \text{a Constant.}$$

This constant is usually called the "gauge factor" and is denoted by the Greek capital letter $\triangle$ (Delta).

$$\text{Hence Gauge Factor} = \triangle = \frac{\delta R / R}{\text{Strain}}$$

a dimensionless constant given by the quotient of ohms per ohm over inches per inch. For nichrome gauge wire the value of $\triangle$ is about 2·25.

Hence from a knowledge of the initial gauge resistance R ohms, the value of the gauge factor $\triangle$, and the measured change of resistance δ R ohms, the magnitude of the strain at the point of application of the gauge can be evaluated from the basic formula :—

$$\text{Strain} = \frac{\delta R}{R} \times \frac{1}{\text{gauge factor}} = \frac{\delta R}{R.\triangle} \text{ inches/inch.}$$

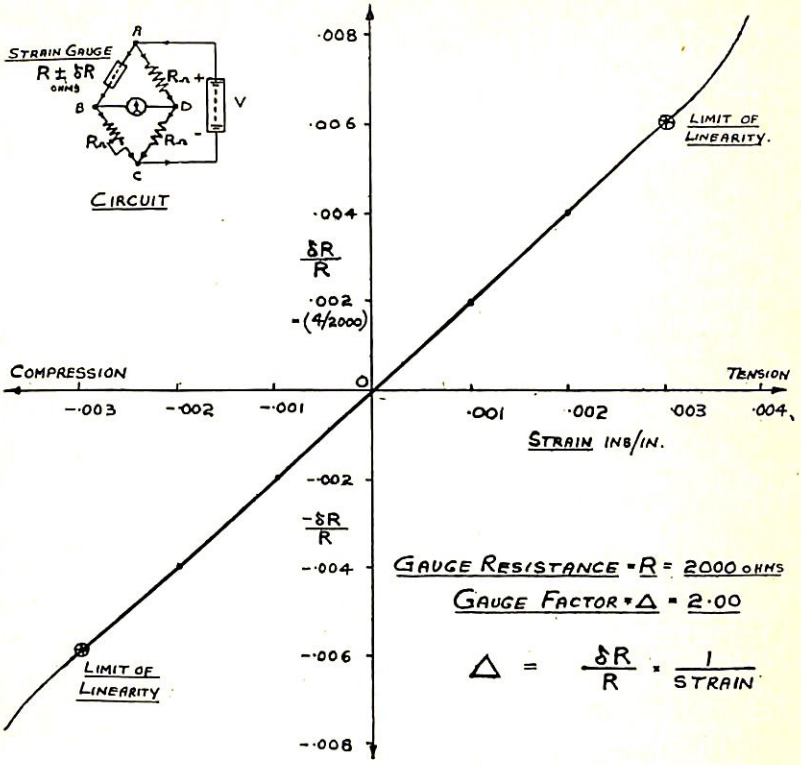


Fig. 2—Determination of Gauge Factor, △

Now the resistance of a given length of wire varies directly as "ρ" the specific resistance in ohms per cm³ and inversely as "a" the cross-sectional area,

$$i.e.,\ R = \frac{\text{Length of Wire}}{\text{Cross Sectional Area}} \times \text{Specific Resistance}$$

$$= l/a \times \rho \text{ ohms.}$$

In the case of a strained wire, however, the cross sectional area depends on the value of Poisson's ratio, *viz.* : $\frac{\text{(Lateral Strain)}}{\text{(Direct Strain)}}$

It can be shown theoretically, that the maximum value of the gauge factor △ cannot exceed $(1 + 2\mu)$—approximately 1·7 (Ref. 10).

It is considered that the higher values of △, determined as a result of experiment, are due to an alteration in the specific resistance of the gauge wire material. The reason for this change in the value of ρ is not yet fully known.

Freak values of $\triangle$ occur; the 5% iridium—platinum alloy has a gauge factor of 5·1, whereas the $2\frac{1}{2}$% chromium—gold alloy has almost zero gauge factor. This latter material is obviously unsuitable for strain-gauge wire in more than one respect.

IV.—CROSS SENSITIVITY.

In every material a direct strain involves a lateral strain equal to $-\mu \times$ direct strain; μ being the value of Poisson's ratio. Thus the semi-circular loops at each end of the single layer type gauge will modify the true value of δ R by an amount depending upon the number of end loops in the strain-gauge.

American manufacturers supply a correction factor with each gauge in order that the true strain may be arrived at.

Gauges of British manufacture are usually of the wound type already described, in which the cross-sensitivity is negligible.

In a biaxial stress field it can be shown that the effect of neglecting the cross-sensitivity is to cause a maximum error in the true strain not exceeding 3%.

V.—MEASUREMENT OF RESISTANCE CHANGE δ R.

The most accurate method of measuring δ R is by means of the Wheatstone bridge*; the galvanometer being used as a null indicator only. In this network, four known resistances are used of which the strain-gauge is one. The resistances are connected up as shown in Fig. 3.

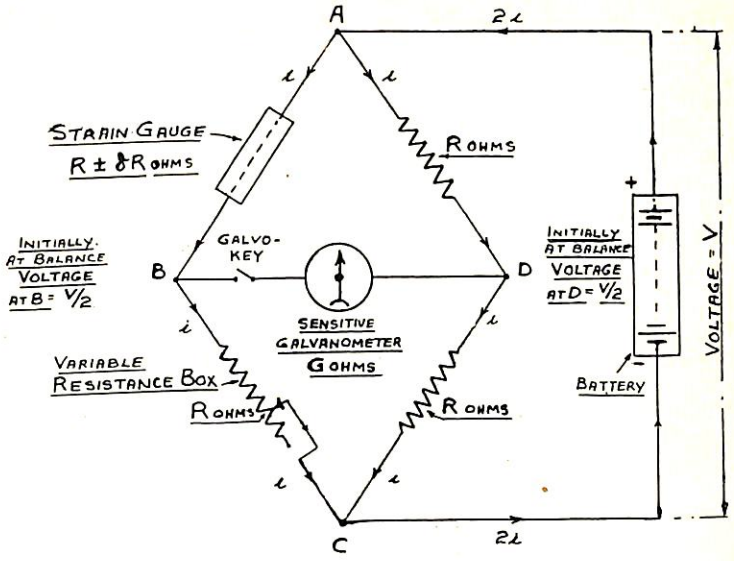
Method 1.—A battery of voltage V is connected across the corners A and C and a sensitive galvanometer is connected across the other two corners B and D. If for simplification the three resistances BC, CD and DA are each made equal to R ohms, the initial value of the strain-gauge resistance, then the battery current will divide equally at A and recombine at C. The potential at B and D will be equal to V/2; hence if the galvanometer key is depressed, no current will flow between B and D, *i.e.*, no potential difference, no current.

Assume now that the gauge AB undergoes positive extension on account of strain in the element of area to which it is attached. The value of R now changes to $R + \delta R$. The potential drop along AB is given by

$$E_{AB} = \frac{\text{Resistance of Arm AB}}{\text{Total Resistance of Arm ABC}} \times \text{Battery Voltage.}$$

$$\text{Or } E_{AB} = V. \frac{R + \delta R}{2R + \delta R} \text{ volts.}$$

*Phil. Trans. Royal Society, 133 (1843), p. 303.



POTENTIAL DIFFERENCE BETWEEN B & D AFTER STRAINING $= \frac{\delta R}{4R}$, V VOLTS

METHOD I

Fig. 3—Wheatstone Bridge.

The potential at D remains constant at V/2 volts.

The potential difference or P.D. between corners B and D is now

$$E_{BD} = V \cdot \frac{R + \delta R}{2R + \delta R} - \frac{V}{2} = V \left\{ \frac{2R + 2\delta R - 2R - \delta R}{4R + 2\delta R} \right\} \text{ or}$$

$$E_{BD} = V \cdot \frac{\delta R}{4R + 2\delta R} \text{ volts.}$$

If δR is small compared with R, as is usually the case, then

$$E_{BD} = V \cdot \frac{\delta R}{4R} \text{ volts} = \textit{Signal Voltage} \text{ (One Active Gauge).}$$

If the galvo-key is now depressed the galvanometer will deflect. In order to reduce the P.D. across corners B and D to zero again, a resistance equal to δR must be inserted in series with the resistance R in arm BC or DA. Inserting δR in arm BC say, then P.D. at B

$$= V. \frac{R + \delta R}{2R + 2\delta R} = \frac{V}{2} \text{ volts.}$$

The value of δR is the figure indicated on the dials of the variable resistance box in arm BC when the deflection of the galvo is again reduced to zero.

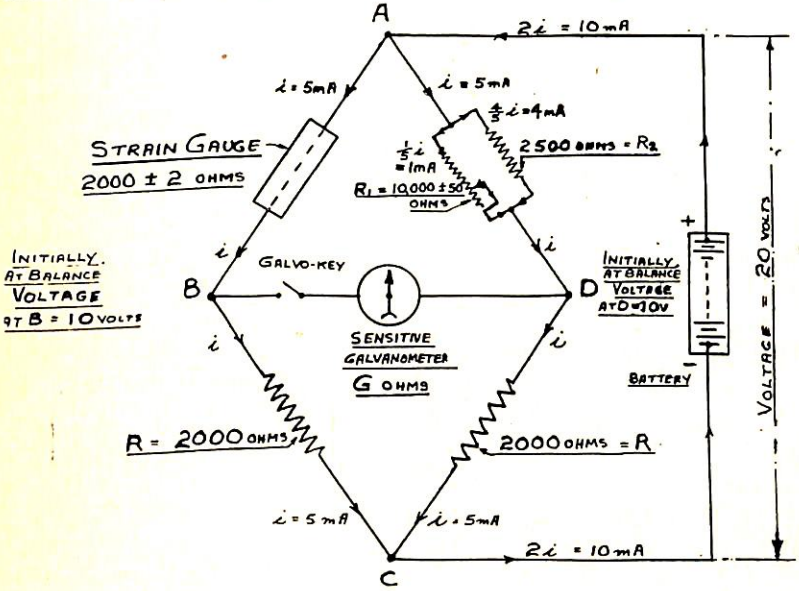
Numerically, if the stress in mild steel is limited to say $6\frac{3}{4}$ tons/in.^{2} corresponding to a strain of ·0005 inches/in., then the change of resistance in a 2000 ohm strain gauge would be, using the basic formula and a gauge factor of 2 :—

$$\delta R = R . \triangle . \text{Strain.}$$

$$\delta R = 2 \times 10^3 . 2 . 1/2 \times 10^{-3} = 2{\cdot}0 \text{ ohms.}$$

If an error not greater than ·1% of the true strain is desired then the error in the measured value of δR must not exceed $\pm$ ·002 ohms.

Since the contact resistance between the sliding surfaces in the best precision resistance box will probably exceed this amount, the above method of measuring δR is not recommended—except for demonstration purposes where accuracy is unimportant.



POTENTIAL DIFFERENCE BETWEEN B & D AFTER STRAINING $= \frac{\delta R}{4 \times 2000} \times 20 = \frac{\delta R}{400}$ VOLTS

METHOD 2.

Fig. 4—Wheatstone Bridge.

Method 2.—A more accurate method is to place a divided circuit of say, 2500 ohms in parallel with a 10,000 ohms resistance, in arm DA of the bridge circuit. This is shown in Fig. 4.

The effective resistance of this combination is derived from the formula $\frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\text{or } R_{eff} = \frac{R_1 R_2}{R_1 + R_2} = 10^3 \left(\frac{10 \times 2{\cdot}5}{10 + 2{\cdot}5}\right) = \frac{25}{12{\cdot}5} \times 10^3$$

$$= 2000 \text{ ohms.}$$

Hence the bridge will be balanced as before, and if the galvanometer key is depressed no deflection will be observed.

For balance, after a change of + 2 ohms in the value of the gauge AB, the resistance to be connected in series with the 10,000 ohm resistance R_1 is determined from the formula :—

$$\frac{1}{R_{AB}} = \frac{1}{R_1^1} + \frac{1}{R_2} \quad \text{or } R_{AB} = \frac{R_1^1 R_2}{R_1^1 + R_2}$$

$$i.e., \ (2000 + 2) = \frac{2500 \times R_1^1}{2500 + R_1^1}$$

Solving for R_1^1 gives a value of 10,050 ohms. Therefore a 50 ohms resistance must be connected in series with the 10,000 ohm resistance R_1 in order to re-balance the bridge.

The contact resistance is now only a small percentage of the measured value and the error introduced is negligible.

Method 3.—An examination of the formula :—

$$\text{Strain} = \frac{\delta R}{R} \times \frac{1}{\text{Gauge Factor}}$$

shows that if the ratio $\frac{\delta R}{R}$ could be measured directly, then absolute values of the initial gauge resistance R ohms and the change in resistance δ R ohms, which is proportional to R, need not be determined.

If the dial of this instrument was graduated in values of $\frac{\delta R}{R}$ %, then the formula for strain would become :—

$$\text{Strain} = \frac{\text{Dial Reading}}{100 \times \text{Gauge Factor}} \text{ ins./in.}$$

The circuit necessary to attain this result is shown in Fig. 5. The contact resistance is now in series with the galvanometer and does not introduce error.

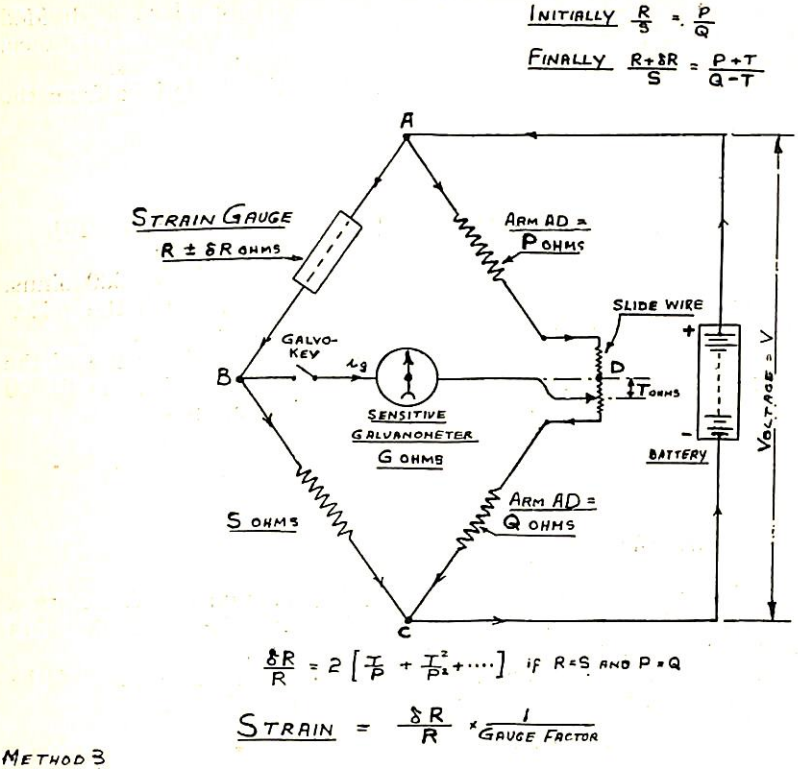


Fig. 5—Precision Slide-Wire Bridge.

Theory—Fig. 5.—At balance T is zero and $\frac{R}{S} = \frac{P}{Q}$

After straining, R changes to $R+\delta R$ and balance is again obtained by adding T ohms to resistance P and subtracting the same amount from resistance Q.

$$\therefore \quad \frac{R+\delta R}{S} = \frac{P+T}{Q-T}$$

$$\text{or} \quad \frac{R}{S} + \frac{\delta R}{S} = \frac{P}{Q} \cdot \frac{(1+T/P)}{(1-T/Q)} \quad \text{and since} \quad \frac{R}{S} = \frac{P}{Q}$$

$$\frac{\delta R}{S} = \frac{P}{Q}\left[\frac{1+T/P}{1-T/Q} - 1\right]$$

$$\text{but } \frac{\delta R}{S} = \frac{\delta R}{R} \times \frac{R}{S} = \frac{\delta R}{R} \times \frac{P}{Q} \text{ since again } \frac{R}{S} = \frac{P}{Q}$$

$$\text{hence } \frac{\delta R}{R} = \frac{1 + T/P}{1 - T/Q} - 1 = \frac{1 + T/P - 1 + T/Q}{1 - T/Q}$$

$$= \frac{\frac{T}{P} + \frac{T}{Q}}{\frac{Q - T}{Q}} = \frac{QT + PT}{PQ} \cdot \frac{Q}{Q - T}$$

$$= \frac{T\,(P + Q)}{P\,(Q - T)}, \text{ which, on dividing out}$$

$$\text{becomes } \frac{\delta R}{R} = \frac{T}{P} + \frac{T}{Q} + \frac{T^2}{PQ} + \frac{T^2}{Q^2} + \ldots.$$

If P is made equal to Q, then R must equal S, and if T is small compared with P then

$$\frac{\delta R}{R} \simeq 2\left(\frac{T}{P} + \frac{T^2}{P^2}\right)$$

Making $P = Q = 100$ ohms and T (Max) $= \pm$ ·25 ohms then the maximum value of $\frac{\delta R}{R}$ that can be measured is $\frac{\delta R}{R}$ max.

$$= \pm 2\left[\frac{1}{4 \times 100} + \frac{1}{16 \times 10{,}000}\right]$$

$$\simeq \pm \frac{1}{200} = \pm \cdot 005 \qquad = \pm \cdot 5 \times \frac{1}{100}$$

$$\text{or } \frac{\delta R}{R} \text{ max} = \pm \cdot 5\%.$$

$$\text{Since Strain} = \frac{\text{Dial Reading}}{100 \times \text{Gauge Factor}} \text{ ins./in.}$$

then max. value, using a gauge factor of 2 is

$$\text{Strain} = \pm \frac{\cdot 5}{100 \times 2} = \pm \cdot 0025 \text{ ins./in.}$$

For steel, inside the limit of proportionality, this strain corresponds to a stress of approximately $\pm$ 34 tons/in.^{2}. The accuracy and range of an instrument of this description makes it ideally suited for static measurements. Method 3 can be easily adapted to record strain at more than one point by adding parallel branches

as shown in Fig. 6. It will be seen that current flows constantly through all the gauges and thus allows a steady temperature to be maintained. This factor largely eliminates zero drift.

Deflection Method.—It might be thought at this stage that a quicker method would be to take readings direct from the galvanometer instead of re-balancing the bridge each time.

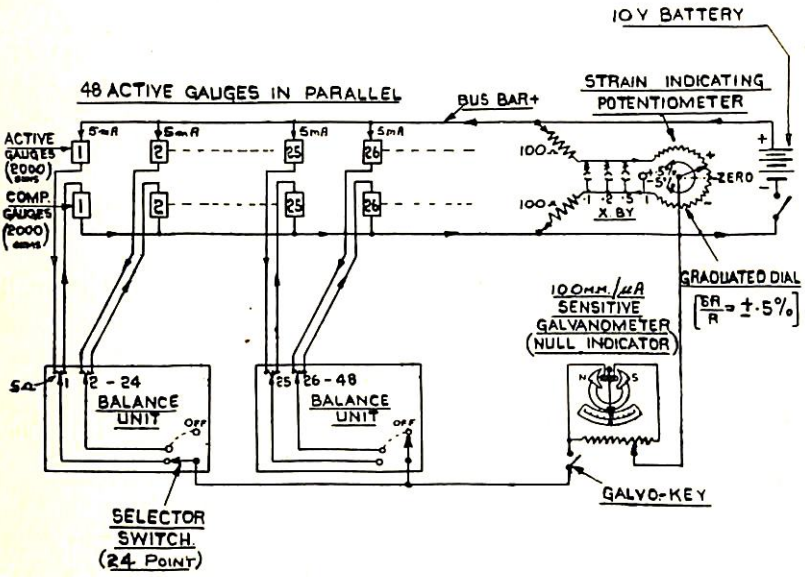


Fig. 6—Multi-Point Strain Measurement Bridge.

Usually, however, the nature of the circuit introduces overdamping of the galvanometer, and appreciable time elapses before the pointer attains a steady deflection. Experience shows that readings can be more quickly taken by the null method. If the slide-wire arm in Figs. 5 and 6 is set at zero, these circuits can be used for the deflection method. Calibration of the galvo scale in terms of strain is effected by varying δ R/R% in suitable steps and noting the resultant deflection on the galvanometer at each step.

The theory of the deflection method is as follows :—

The galvanometer current i_g, for a given change in resistance δ R ohms and a gauge current i, can be shown to be, (Thévenin's theorem),

$$i_g = \frac{i \cdot \delta R}{G + \left(\frac{(R+P)\ (Q+S)}{R+P+Q+S}\right)} \times \frac{(Q+S)}{R+P+Q+S}$$

$$\text{If } \frac{R}{S} = \frac{P}{Q} = 1, \text{ then } i_g = \frac{i \cdot \frac{\delta R}{R}}{2\frac{G}{R} + \frac{P}{R} + 1} \quad \text{(Fig. 5).}$$

An examination of the denominator shows that if P is made small compared with R, maximum sensitivity will be attained. To avoid a heavy battery current and consequent drop in voltage, the value of R is usually 20 to 100 times the value of P. Optimum sensitivity, for which $i_g^2 \cdot G$ is a maximum, occurs when the galvanometer resistance

$$G = \frac{(R+P)\ (Q+S)}{R+P+Q+S} = \frac{(P+R)^2}{2\ (P+R)} = \frac{P+R}{2} \text{ if } \frac{R}{S} = \frac{P}{Q} = 1$$

If the arms P and Q of the bridge circuit were made very small, then the best value of the galvanometer resistance would be G = R/2 ohms. Some sacrifice of sensitivity may be necessary in order to avoid overdamping or sluggish operation of the galvo.

It will be obvious from the formula for i_g, that when a deflection method is used the resistance G of the galvanometer *plus its attendant leads*, must be kept constant. Changes in ambient temperature affect the value of G and introduce considerable error.

VI.—TEMPERATURE COMPENSATION.

Up to now the effect of change in temperature has not been considered. If at balance the strain-gauge attached to the test surface experiences a change in temperature, then the combination of (a) the linear expansion of the test surface, adhesive and paper, and (b) the temperature coefficient of resistance of the strain-gauge wire (·004 ohms per ohm per°C. for copper) would result in a change of gauge resistance say δR^T. This change would, as previously explained, unbalance the bridge and be measured as an apparent strain. Fictitious readings of this character must be eliminated at all costs, if accuracy of results is to be maintained.

Temperature effects can be automatically nullified by utilising another strain-gauge, identical to the gauge on the test surface as a "temperature compensator."

This gauge must be placed in series with the "active" gauge R and thus replaces the resistance in arm BC of the bridge.

In addition it must be affixed to an *unstressed* piece of the same material as that under test, and should be placed as near as possible to the active gauge.

Both gauges will now experience the same temperature change, and in consequence, equal variation in resistance δR^T. Hence unbalance of the bridge cannot occur, since the same amount of resistance has been placed in each of the adjacent arms AB and BC.

In simple bending tests the compensating gauge may be affixed immediately opposite the active gauge, provided the test object is symmetrical about its neutral axis. By this means perfect temperature compensation is achieved and the sensitivity of the bridge is doubled.

VII.—ZERO DRIFT AND CREEP.

After a bridge has been balanced or zeroised, it is sometimes observed that a slow persistent drift of the galvo-pointer away from zero takes place. If it is known that tensile strains cause the pointer to deflect to the right say, and the drift is observed to be in the opposite direction, then this phenomenon, usually called zero drift, is due to faulty insulation between the gauge wires and earth. This is generally due to absorption of moisture by the gauge from the atmosphere.

Zero drift can be satisfactorily eliminated by ensuring that insulation to earth is never less than 50 megohms.

Poor insulation is equivalent to shunting the gauge with a high resistance, *e.g.*, if a 2500 ohm gauge is shunted by a resistance of 3·3 megohms, then the equivalent decrease in gauge resistance is approximately 2 ohms. This corresponds to an apparent compressive stress of 5 tons/in.^{2} in steel.

For static measurements the ideal of infinite resistance should always be aimed at.

A moisture-resistant coating should be applied to the gauge after the glue solvent has dried out and the insulation to earth has been tested by means of an electrician's "megger." A quick-drying transparent liquid plastic, possessing water and oil resistant properties is marketed by I.C.I., Ltd. (Welwyn) under the name of "Tensol," No. 2.

Random drifts or quick excursions of the galvo-pointer are usually caused by sudden draughts, radiation from adjacent objects or direct sunlight. American manufacturers usually supply a felt pad with each gauge in order to eliminate this particular error. Random drifts can also be traced to intermittent contact at plug points and terminals. Soldered joints should be made wherever possible. This last statement cannot be over-emphasised.

Creep is another type of drift which is observed after the gauge has been strained by a fixed amount. In this case the drift is

always towards zero ; the source of trouble is usually unsatisfactory adhesive, faulty sticking technique or insufficient drying-out of the adhesive solvent.

VIII.—MEASUREMENT OF STEADY TORQUE.

In order to measure pure torque it is first necessary to determine the orientation of the gauges with respect to the axis of the shaft. If we consider an element of surface area, with unit side and thickness, subjected to a shear stress q acting along $\overline{AB}$ and $\overline{CD}$, Fig. 7a, then equilibrium conditions show that an identical shear stress must appear along the sides $\overline{AD}$ and $\overline{CB}$. This is known as complementary shear (Fig. 7b). Resolving the shear *forces* along and perpendicular to the diagonal DB, it will be seen that a force of $\sqrt{2}.q$ acts in a direction perpendicular to the diagonal AC, Fig, 7c. The length of AC is $\sqrt{2}$, therefore :—

$$\text{Normal Stress} = \frac{\text{Load}}{\text{Area}} = \frac{\sqrt{2}.q}{\sqrt{2}} = q = f_x \text{ say.}$$

Similarly a normal stress of $-q$ acts on diagonal BD $= -f_y$ say. Hence a state of pure shear is equivalent to the action of two equal mutually perpendicular principal stresses of opposite sign acting

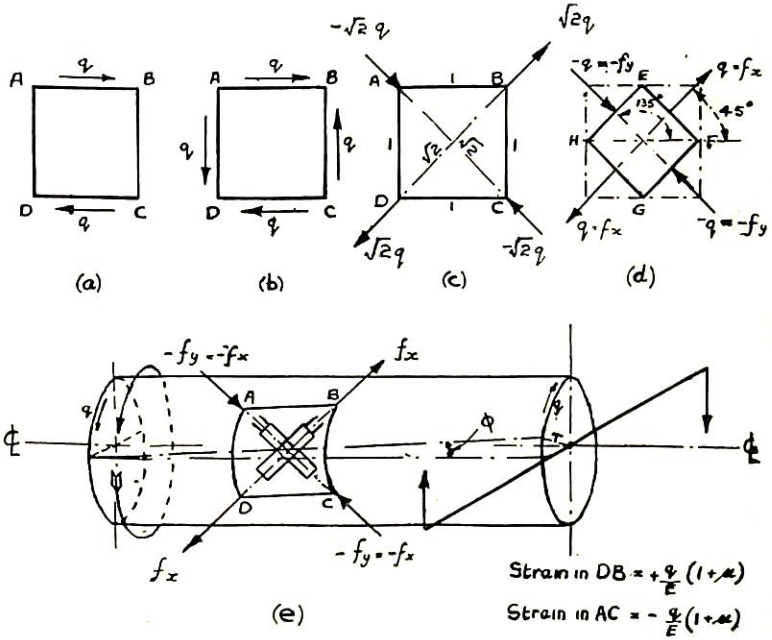


Fig. 7—Determination of Normal Stresses in Shaft subjected to Torsion.

in a direction inclined at 45° and 135° respectively to the plane of shear, Fig. 7d. If two sets of strain gauges are placed on opposite sides of the shaft, one set being shown in Fig. 7e, then application of a pure torque will result in an extension of two of the strain gauges, and a contraction of the other two. This is caused by the principal stresses f_x and $-f_y$ acting at 45° and 135° respectively to the shaft axis.

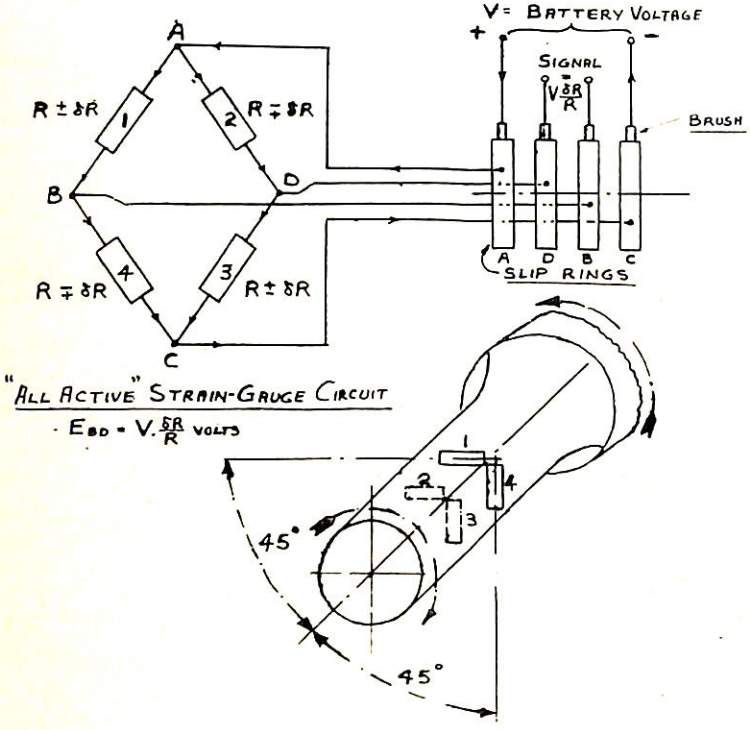


Fig. 8—Measurement of Torque in Shaft.

The gauges are connected up as shown in Fig. 8. In this case all four gauges are active. By this method of connection perfect temperature compensation is effected and the output voltage across the corners B and D is four times that of a similar circuit containing only one active gauge, *i.e.*, $E_{BD} = V \cdot \delta R/R$ volts. It will be noted, that in the case of a torsion meter the shear stress is used as a measure of the applied torque. In order to read the torque directly, a deflection method is used. The galvanometer scale is calibrated in terms of torque by applying known torques to the shaft and noting the corresponding deflections.

IX.—TORSION COMBINED WITH BENDING AND AXIAL LOAD.

The arrangement of gauges as shown in Fig. 8 is responsive to pure torque only. For instance, if bending occurs say concave downwards, then the two gauges AB and BC on the top surface of the shaft will experience an increase of resistance $+ \delta R_B$ whereas gauges CD and DA on the bottom surface of the shaft will suffer a decrease of resistance $- \delta R_B$.

Now resistance of arm ABC $= 2R + 2\delta R_B$

and resistance of arm ADC $= 2R - 2\delta R_B$

$$\therefore \text{Voltage at B} = \frac{R + \delta R_B}{2R + 2\delta R_B} \cdot V = \frac{V}{2} \text{ volts.}$$

$$\text{Similarly voltage at D} = \frac{R - \delta R}{2R - 2\delta R} \cdot V = \frac{V}{2} \text{ volts.}$$

Thus the circuit is not responsive to strain produced by bending. Slight response may occur, however, if the gauges are not correctly orientated.

Similarly if the shaft is subjected to axial load, the resistance of all four gauges will change to $R \pm \delta R_A$. By the same reasoning as before, it will be seen that no potential difference will appear between the terminals B and D of the bridge circuit.

X.—DYNAMIC LOADING—FLUCTUATING TORQUE.

For rotating shafts, each corner of the bridge A, B, C and D is connected to a separate slip ring. All four rings rotate with the shaft but are insulated from it. The material is usually stainless steel. For precision measurements pure silver is used.

Contact with the slip rings is made by silver-loaded graphite brushes—3 per set—spaced at 120° intervals around the shaft. The voltage is applied across the two outer rings connected to A and C respectively, and the signal voltage produced by the fluctuating torque is picked up by the two centre sets of brushes—Fig. 8. A general view of such a torsionmeter is shown in Fig. 8a.

Contact-resistance errors may be reduced to a small percentage if a large resistance is connected in series with the battery lead. In this case the voltage applied to the bridge must be measured across the slip rings.

The current through the gauges is fixed by the thermal stability of the insulation. If the current is large, heat dissipation is variable and a uniform steady temperature is difficult to maintain. This leads to the appearance of fictitious voltages at the output terminals, *i.e.*, slip rings B and D.

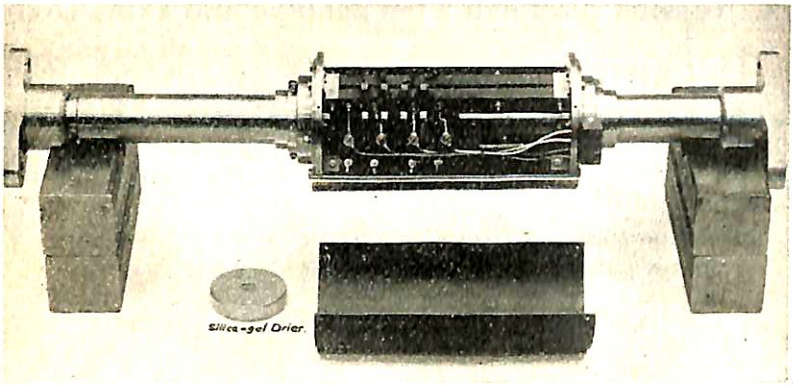


Fig. 8a.

Torsionmeter: (500 lb. ft. max.) Output Signal = 0.112 m.V. per lb. ft.

On this account the current is usually limited to ·005 amperes, *i.e.*, 5 milliamps. Hence the applied voltage must be $2iR = ·005 \times 2000 \times 2 = 20$ volts. With this steady pressure of 20 volts applied to the all active bridge, Fig. 8 (2000 ohms per arm), the output or signal voltage for a change of ±2·0 ohms per 2000 ohm resistance will be :—

$$E_{BD} = \frac{\delta R}{R} \cdot V \text{ (since 4 gauges are active).}$$

$$\text{Or } E_{BD} = \frac{2}{2000} \times 20 = ·020 \text{ volts or 20 millivolts.}$$

A *steady* signal voltage of this magnitude would actuate a delicate, centre zero, moving coil micro-ammeter. However, on account of the inherent inertia of the moving-coil assembly, the instrument would not respond to the instantaneous values of the signal voltage sent out by the strain gauges on the rapidly flexing shaft.

In such a circumstance the reading of the galvanometer or micro-ammeter would represent the mean value of the applied torque.

If a permanent record of the torque variation is required, a pen recorder or a cathode ray oscillograph and camera must be used.

XI.—CATHODE RAY OSCILLOGRAPH (C.R.O.).

It is hoped to describe this instrument in greater detail in a companion booklet on strain-gauge recording equipment. There, it will be seen, that a fine pencil of rapidly moving electrons is

focussed and constrained to impinge on a fluorescent screen thus causing a spot of light to appear. Since the electron represents only a small fraction of the weight of the atom from which it has been freed, it will be appreciated that the cathode ray or pencil of electrons is equivalent to the pointer of an instrument with an almost inertia-less moving system—the dream instrument of the Victorian era physicist.

If now two metal plates are arranged, condenser fashion, on either side of the electron beam, and a potential difference applied to them, then by virtue of the electrostatic attraction and repulsion between the charged plates and the negatively charged electron beam passing between them, the spot of light on the fluorescent screen will deflect in direct proportion to the instantaneous voltage across the two plates.

The minimum value of the voltage necessary to deflect the electron beam is of the order of 100 volts. In consequence the signal voltage of ·020 volts (20 millivolts) must be amplified about 5000 to 8000 times.

The direct-current amplifier, *i.e.*, one that responds accurately to slow changes of input voltage, has not at present attained the same degree of "zero" and [redacted] gain stability as the ordinary A.C. resistance-capacitance amplifier. This latter amplifier only responds to rapid changes in voltage (Ref. 8).

Unfortunately the A.C. amplifier,* unless of very good design, may have a non-linear input/output relationship caused by any or all of the four following defects :—

(*a*) Amplitude distortion, *i.e.*, for a *variable* input voltage at a *fixed* frequency the output/input ratio may not remain constant.

(*b*) Frequency distortion, *i.e.*, for a *fixed* input voltage supplied at *different* frequencies, the output/input ratio may vary with each frequency.

(*c*) Phase distortion, *i.e.*, for a *fixed* input voltage supplied at *different* frequencies the output/input phase relationship may be different for each frequency ; hence the output wave form will not be a replica of the input voltage wave form.

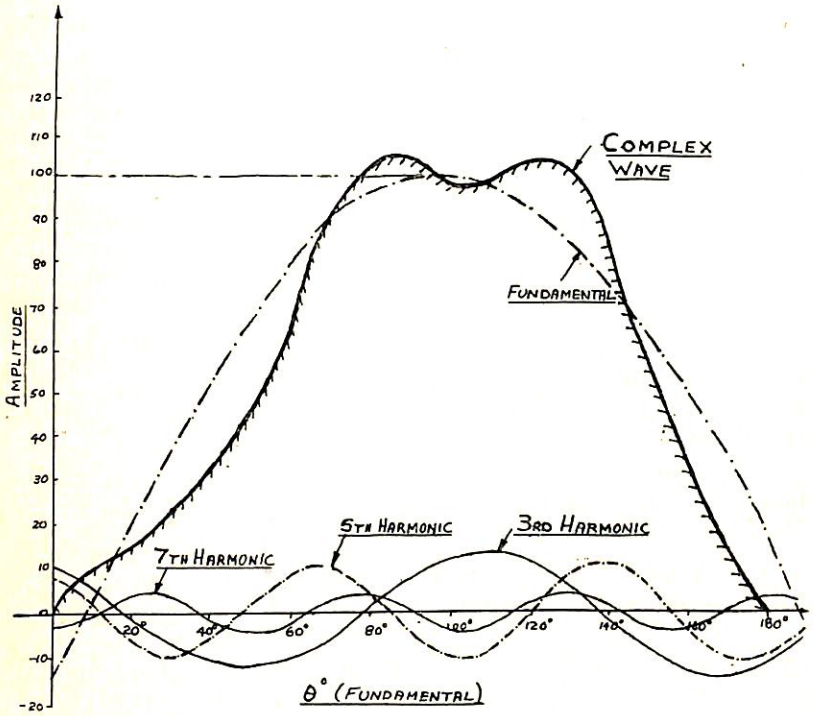
(*d*) Noise, hum, microphony and "pickup," *i.e.*, external signals may be added to the input signal.

(*a*) and (*d*) can usually be reduced to small proportions. Of (*b*) and (*c*) the latter is the most serious, especially when slow transient phenomena are to be recorded.

* "Radio Engineering," by F. E. Terman. (McGraw-Hill Book Co., N.Y., 1937).

It may be queried at this juncture why frequency and phase relationship need to be considered at all, since inspection of a typical transient record shows a trace consisting of single non-sinusoidal hump or half-wave with an appreciable time duration (Fig. 9).

To explain this it is necessary to refer to Fourier's theorem and harmonic analysis (Ref. 9).



EQUATION OF WAVE = $100 \sin(\theta - 8^\circ) + 13 \sin(3\theta + 126^\circ) + 10 \sin(5\theta + 120^\circ) - 4 \sin(7\theta + 84^\circ)$

Fig. 9—Harmonic Analysis of Complex Wave.

XII.—FOURIER SERIES.

The method of representing the function of a single independent variable that repeats at regular intervals, by an infinite series of sines and cosines of multiple angles was stated by Fourier (1820). Such functions are known as "periodic" and satisfy the relation $f(x) = f(x + \alpha)$ where α is a constant known as the "period." For example we know that $\sin x$, $\cos x$, $\tan x$ and their reciprocals satisfy for all values of x the relation $f(x) = f(x + 2\pi)$ where 2π is the period or "periodic time."

By this theorem it is shown that a saw tooth wave represented by the explicit function :—

$y = f(x) = x$ between $x = 0$ and $x = 2\pi$ can also be represented by the infinite series $y = \pi - 2 (\sin x + \frac{1}{2} . \sin 2x + \frac{1}{3} . \sin 3x + \frac{1}{4} . \sin 4x + \ldots)$, a series comprising the fundamental $\sin x$, and odd and even harmonics.

Similarly the square wave represented by :—

$y = 0$ from $x = 0$ to $x = \pi$ and $y = 1$ from $x = \pi$ to $x = 2\pi$ may also be represented by $y = \frac{1}{2} - 2/\pi (\sin x + \frac{1}{3} . \sin 3x + \frac{1}{5} . \sin 5x + \frac{1}{7} . \sin 7x + \ldots)$, a series in which only odd harmonics are present.

Fig. 10 shows how the sum of the first six terms of the series representing the function $y = x$ (from $x = 0$ to $x = 2\pi$), approaches its ultimate linear form of a straight line through the origin inclined at 45° to the horizontal axis.

A transient in the form of a graph can be said to be a periodic function that is not known explicitly.

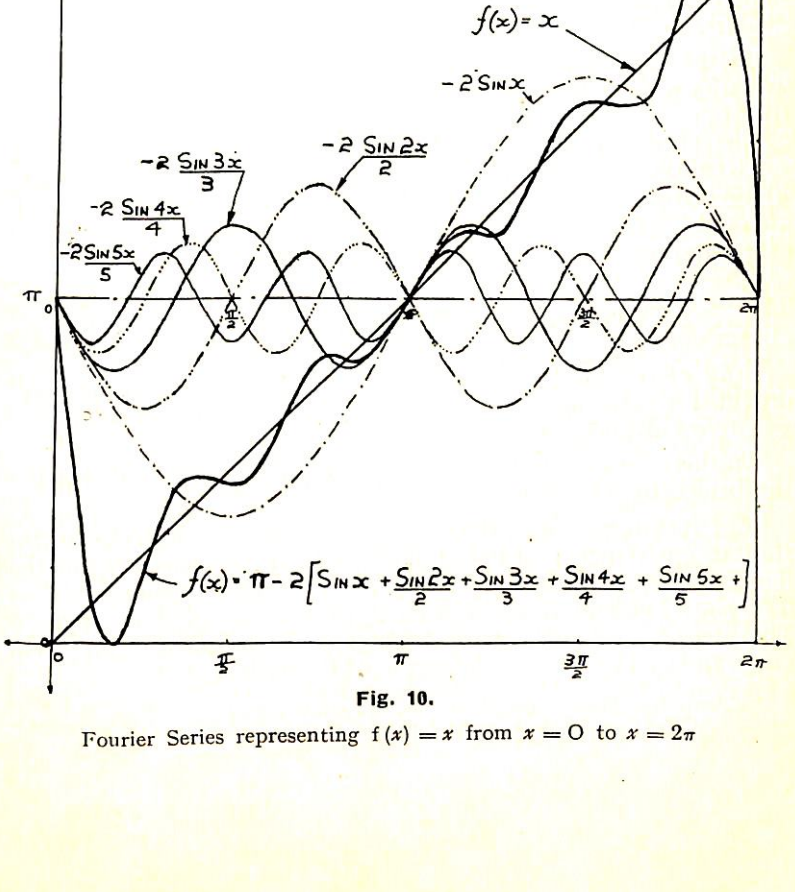


Fig. 10.

Fourier Series representing $f(x) = x$ from $x = 0$ to $x = 2\pi$

It is possible to analyse or "split up" a complex voltage wave, and express the result as a Fourier series of sinusoidal terms. It will thus be evident that to apply a transient wave of voltage to the input terminals of an A.C. amplifier is equivalent to feeding in, simultaneously, an infinite number of sinusoidal voltage waves, each with a different frequency and amplitude. Fig. 9 shows the analysis of a complex wave.

If the input voltage wave is represented by $e = \sum_{n=1}^{n=\infty} E_n \cdot \text{Sin}$ $(n\,wt \pm \theta_n)$ where n is the order of the harmonics 1, 2, 3, 4 etc., and θ is the phase angle with respect to the initial line, then the output may be of the form $e = \sum_{n=1}^{n=\infty} E'_n \cdot \text{Sin}\ (n\,w\,t \pm \theta_n \pm \beta_n)$, where $w = 2\pi \times$ frequency.

This shows that for each individual wave, the output amplitude is not independent of the frequency, and that there is a phase displacement β and hence delay time $d\beta/dw$ between each component input and output wave. The angle β and the delay time is large and most apparent in the low frequency components of the signal, on account of the increased reactance of the coupling condensers.

If the distortion is appreciable then the summation or output wave form as delineated on the photographic record will not be a true reproduction of the stress wave at the point of application of the strain-gauge. For slow transients the A.C. modulated carrier-wave method eliminates frequency and phase distortion. Phase distortion may be observed by noting the fidelity with which an amplifier can reproduce a square wave. This test is recognised as a criterion of performance.*

XIII.—DYNAMIC MEASUREMENTS.

Any of the foregoing circuits may be used for dynamic work provided a cathode ray tube and amplifier is used instead of the sensitive galvanometer.

In this case a calibration or step record is taken by unbalancing the bridge by a known amount.

For instance, if the 10,000 ohm resistance in Fig. 4 was instantaneously increased by 50 ohms, then the displacement of the fluorescent spot on the C.R.O. as a result of the suddenly produced unbalanced voltage, would represent a tensile stress of $6\frac{3}{4}$ tons/in.^{2}. The step record is the trace registered on the moving film of a camera placed in front of the screen of the cathode ray tube.

* "Amplifier Testing by Means of Square Waves."
(Swift, Communications, p. 22, Feb., 1939).

From what has already been written on the difficulties encountered in recording fluctuating signals, it will be obvious to the mechanical engineer, interested in stress alone, that dynamic measurements require (*a*) a knowledge of fundamental electronics, and (*b*) operational skill in adjusting and maintaining the equipment if grievous error is to be avoided. On this account, strain gauge work has tended to become a specialist branch of engineering research.

XIV.—ACCURACY OF RESULTS.

Since the strain in the test surface is transmitted to the strain-sensitive wire through the medium of an adhesive, it is obvious that the ratio of the observed strain to the true strain will always be slightly less than unity.

Other possible sources of error can be attributed to (*a*) variation in gauge factor; (*b*) inadequate temperature compensation; (*c*) variation in the applied voltage; (*d*) variation of the calibration resistances with time; and (*e*) variation of amplifier gain.

By a system of quality control the error in the gauge factor can be guaranteed not to exceed $\pm$ 3%.

This is achieved by calibrating one gauge per batch on a simple cantilever, or on a beam simply supported at the first and third quarter-points and loaded equally at each end.

Adherence to a good standard of electrical practice and a high level of instrument maintenance will largely eliminate errors due to (*c*), (*d*) and (*e*).

In static work, with all possible precautions taken, the error in the observed strain should not be more than $\pm$ 5%.

For dynamic measurements the error in a single observation would be of the order of $\pm$ 10%. Comparative readings could reduce this figure to about $\pm$ 5%.

Although the accuracy of results obtained by the electrical strain-gauge method is less than the figure obtainable from mechanical and optical extensometers, it should be remembered that the electrical strain-gauge is usually only employed in situations unsuitable for or impossible of access to the orthodox type, and on structures where the vibratory nature of the load makes the use of any other type of gauge impossible.

XV.—DERIVATION OF PRINCIPAL STRESSES (STRESS ANALYSIS).

The value of the stress determined from the reading of a single strain-gauge placed at random on a test surface will not, in general, give the maximum possible value; unless, of course, the gauge has been fortuitously placed along the line of action of a principal stress.

In the case of axial and biaxial stress such as obtains in a simple tension member and on the surface of a boiler shell respectively, the direction of each principal stress is known and orientation of the strain gauges presents no difficulty.

Where the stress is of a complex nature, *i.e.*, shear stress is present in addition to a normal stress, some method of determining the magnitude, sense and direction of each principal stress must be devised (Refs. 1 and 2).

Complex Stress.—In a two-dimensional field of complex stress it will be found that there exist two mutually perpendicular planes on each of which the resultant stress is a NORMAL STRESS ONLY, *i.e.*, no shear stress is present. These two mutually perpendicular planes are called principal planes and the normal stress acting on each is called a principal stress.

The algebraically greater, and the algebraically smaller of the two principal stresses, are known as the maximum principal stress and the minimum principal stress respectively.

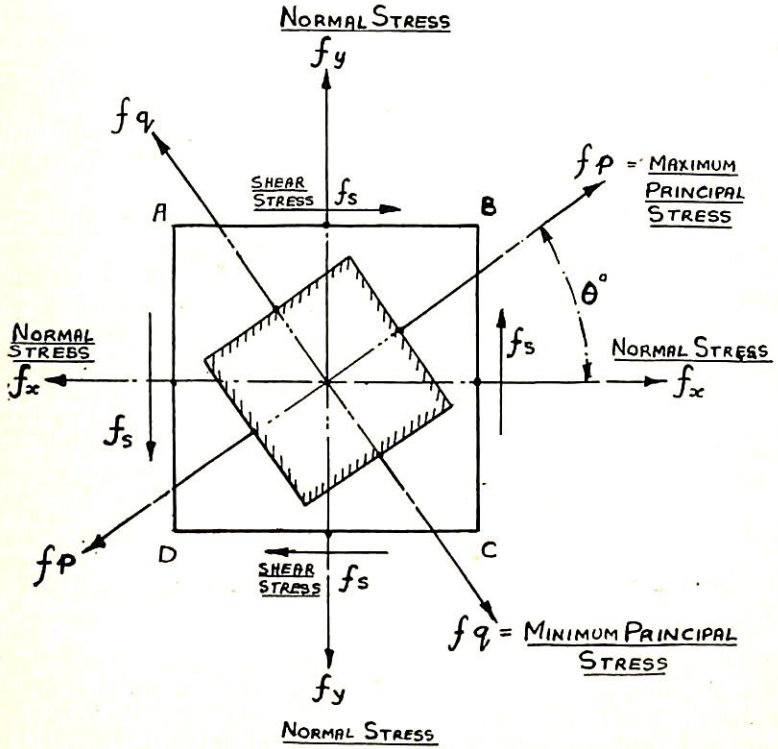


Fig. 11—Normal, Shear and Principal Stresses.

In Fig. 11, ABCD represents an element of area of unit side and thickness, subjected to a direct tensile stress of magnitude f_y on the sides AB and CD, and a direct tensile stress of magnitude f_x on the other two sides AD and BC. Each of the four sides is also subject to a shear stress of magnitude f_s.

The problem is, assuming the above set of conditions, to find the maximum and minimum principal stress, f_p and f_q respectively, and θ the angle between the direction of the maximum principal stress f_p and the direction of the normal stress f_x.

In order to do this, measurements of strain are made in three directions, *i.e.*, two measurements are made at right angles to each other and a third along the line bisecting this right angle. The orientation of the three lines, treated as a unit, is chosen arbitrarily. From the geometry of the three directions of measurement, this particular formation of lines is known as the rectangular strain rosette—Fig. 12b.

Before Mohr's circle of strain and stress can be applied to the solution of a rectangular rosette, the strain along a line drawn arbitrarily in a field of complex stress, must be calculated. In Fig. 13, let AB be a line inclined at an angle θ degrees to the direction of the strain e_x. It is required to find the strain in the line AB.

The extension of CB due to the longitudinal strain $e_x = BB' = CB \cdot e_x$.

DB', the change in length of $AB = BB' \cdot \text{Cos}\ \theta = CB \cdot e_x \cdot \text{Cos}\ \theta$.

Hence the strain in AB $= \dfrac{CB \cdot e_x \cdot \text{Cos}\ \theta}{AB} = \dfrac{CB \cdot e_x \cdot \text{Cos}\ \theta}{CB/\text{Cos}\ \theta}$

$= e_x \cdot \text{Cos}^2\theta$.

By similar reasoning the strain in AB due to a strain e_y in a direction perpendicular to e_x would be $e_y \cdot \text{Sin}^2\theta$.

Therefore the total strain due to e_x and e_y acting together is equal to

$$e_{AB} = e_x \cdot \text{Cos}^2\theta + e_y \cdot \text{Sin}^2\theta \text{ and since } \text{Cos}^2\theta = \frac{1 + \text{Cos}\ 2\theta}{2}$$

and $\text{Sin}^2\theta = \dfrac{1 - \text{Cos}\ 2\theta}{2}$, then

$$e_{AB} = \frac{(e_x + e_y)}{2} + \frac{(e_x - e_y)}{2} \cdot \text{Cos}\ 2\theta.$$

Having found the strain in AB due to strains e_x and e_y, the additional strain caused by the action of the shear strain ϕ on the element must next be determined.

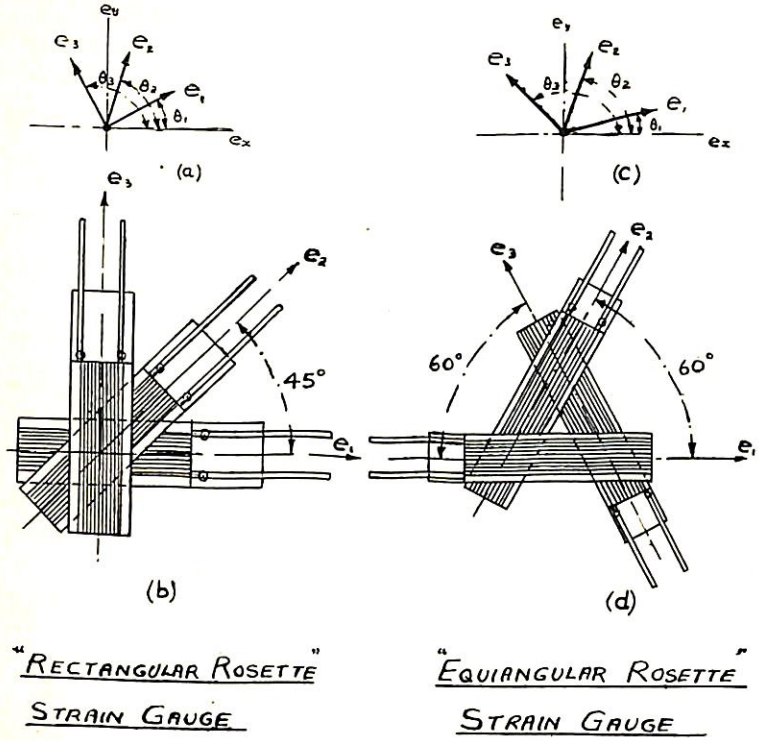
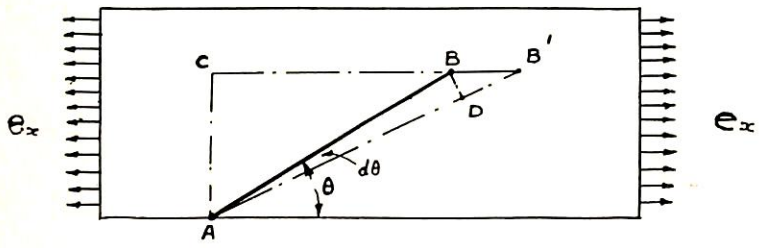


Fig. 12—Rosette Type Strain Gauges.



EXTENSION $BB' = CB \times \text{STRAIN} = CB \cdot e_x$

Fig. 13—Strain in AB due to Longitudinal Strain e_x

In Fig. 14 $BB' = BD \text{ Tan } \phi = BD \,.\, \phi$ since ϕ the shear strain is small.

and $B'C = BB' \text{ Cos } \theta = BD \,.\, \phi \,.\, \text{Cos } \theta$

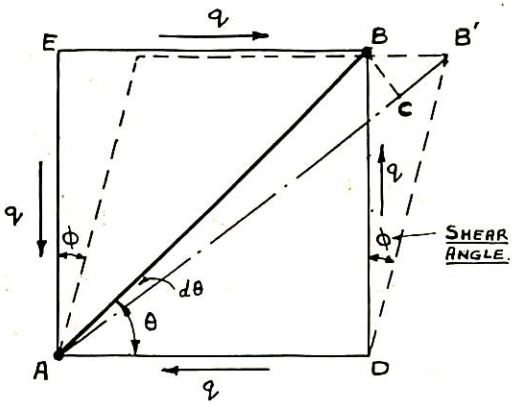
then $e_{AB} = \dfrac{B'C}{AC} = \dfrac{B'C}{AB} = \dfrac{BD \,.\, \phi \,.\, \text{Cos } \theta}{BD/\text{Sin } \theta}$

$= \phi \,.\, \text{Sin } \theta. \text{ Cos } \theta = \phi/2 \,.\, \text{Sin } 2\theta.$

Therefore the resultant strain in AB $= e_\theta$ say, due to the combined action of e_x, e_y and ϕ is

$$e_\theta = e_x \,.\, \text{Cos}^2\theta + e_y \,.\, \text{Sin}^2\theta + \phi \,.\, \text{Sin } \theta \,.\, \text{Cos } \theta.$$

$$\text{Or } e_\theta = \frac{e_x + e_y}{2} + \frac{(e_x - e_y)}{2} \cdot \text{Cos } 2\theta + \frac{\phi}{2} \cdot \text{Sin } 2\theta.$$



EXTENSION $BB' = BD.\text{TAN } \phi = BD.\phi$ SINCE ϕ IS SMALL

Fig. 14—Strain in AB due to Shear Strain ϕ

There will be some value of θ for which the strain e_θ will be a maximum. Hence differentiating the above expression and equating to zero we get :—

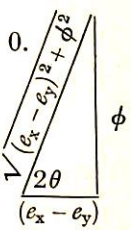
$$d/d\,\theta\,(e_\theta) = e_x \,.\, 2 \text{ Cos } \theta \,.\, -\text{Sin } \theta + e_y \,.\, 2 \text{ Sin } \theta \,.\, \text{Cos } \theta + \phi/2 \,.\, 2 \text{ Cos } 2\theta = 0.$$

$$= -\, e_x \,.\, \text{Sin } 2\theta + e_y \,.\, \text{Sin } 2\theta + \phi \,.\, \text{Cos } 2\theta = 0.$$

On dividing by Sin 2θ we get : $-\, e_x + e_y + \dfrac{\phi}{\text{Tan } 2\theta} = 0.$

$$\text{Or Tan } 2\theta = \frac{\phi}{e_x - e_y}$$

$$\therefore \; 2\theta = \left(n\pi + \text{Tan}^{-1} \frac{\phi}{e_x - e_y} \right) \text{ where } n \text{ is zero}$$



or an integer 1, 2, 3, etc., *i.e.*, there are two values of 2θ differing by 180°.

Substituting the value of Cos 2θ and Sin 2θ respectively in the formula for e_θ, then

$$e_{\theta\,\text{max.}} = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cdot \frac{e_x - e_y}{\sqrt{(e_x - e_y)^2 + \phi^2}} + \frac{\phi}{2} \cdot \frac{\phi}{\sqrt{(e_x - e_y)^2 + \phi^2}}$$

$$= \frac{e_x + e_y}{2} + \frac{(e_x - e_y)^2 + \phi^2}{2\sqrt{(e_x - e_y)^2 + \phi^2}}$$

$$e_{\theta\,(\text{max.})} = \frac{e_x + e_y}{2} + \frac{1}{2}\sqrt{(e_x - e_y)^2 + \phi^2} = \text{Max. Principal Strain} = e_p$$

$$e_{\theta\,(\text{min.})} = \frac{e_x + e_y}{2} - \frac{1}{2}\sqrt{(e_x - e_y)^2 + \phi^2} = \text{Min. Principal Strain} = e_q$$

In order to determine e_x, e_y, ϕ and hence the two values of e_θ, three strain-gauges are affixed to the test surface in the formation known as a rectangular rosette.

In Fig. 12 (*a*) if $\theta_1 = 0°$, $\theta_2 = 45°$ and $\theta_3 = 90°$; then using the formula :—

$$e_\theta = e_x \cdot \text{Cos}^2\theta + e_y \cdot \text{Sin}^2\theta + \phi \cdot \text{Sin}\ \theta \cdot \text{Cos}\ \theta \cdot$$

and substituting θ_1, θ_2, θ_3.

we have $e_1 = e_x \cdot 1 + 0 + 0.$

$e_3 = 0 + e_y \cdot 1 + 0.$

$e_2 = e_x \cdot \frac{1}{2} + e_y \cdot \frac{1}{2} + \phi \cdot \frac{1}{2}.$

i.e., $e_1 = e_x$, $e_3 = e_y$, and $e_2 = e_1/2 + e_3/2 + \phi/2$.

Hence $e_x = e_1$.

$e_y = e_3$.

and $\phi = 2e_2 - e_1 - e_3$, and on substitution in the formula for e_θ max.
min.

then $e_{pq} = \frac{e_1 + e_3}{2} \pm \sqrt{\left(\frac{(e_1 - e_3)}{2}\right)^2 + \left(\frac{2e_2 - e_1 - e_3}{2}\right)^2}$

and Tan $2\theta_{pq} = \frac{2e_2 - e_1 - e_3}{e_1 - e_3}$, the angle 2θ being measured C.C.W. from the direction of e_p.

If we let $\frac{e_1 + e_3}{2} = A$ and $\sqrt{\left(\frac{e_1 - e_3}{2}\right)^2 + \left(\frac{2e_2 + e_1 - e_3}{2}\right)^2} = B$, then $e_{pq} = A \pm B$.

This formula can be represented graphically by a circle of radius B units with its centre on the x axis, A units from the origin. The construction is shown in Fig. 15.

The two principal strains e_p and e_q respectively can now be used to evaluate the two principal stresses f_p and f_q as follows :—

$$\text{If } \mu = \text{Poisson's Ratio then } e_p = \frac{f_p}{E} - \frac{\mu f_q}{E} \quad (1)$$

$$\text{and } e_q = \frac{f_q}{E} - \frac{\mu f_p}{E} \quad (2)$$

Substituting the value of f_q in terms of f_p from equation (2) in equation (1), we get :—

$$\text{Max. Principal Stress } f_p = \frac{E}{1-\mu^2}(e_p + \mu e_q)$$

$$\text{and Min. Principal Stress } f_q = \frac{E}{1-\mu^2}(e_q + \mu e_p)$$

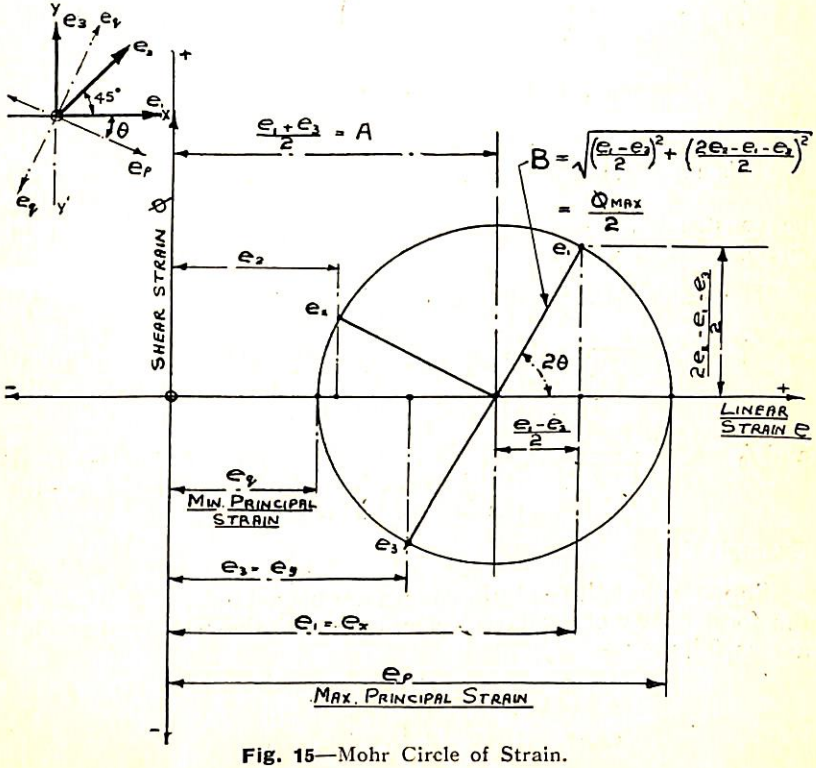


Fig. 15—Mohr Circle of Strain.

Now $e_p = A + B$ and $e_q = A - B$.

$$\therefore f_p = \frac{E}{1-\mu^2}\left((A+B) + \mu(A-B)\right)$$

$$f_p = \frac{E}{1-\mu^2}\left(A(1+\mu) + B(1-\mu)\right)$$

$$f_p = E\left(\frac{A}{1-\mu} + \frac{B}{1+\mu}\right) = \text{Max. Principal Stress.}$$

and similarly

$$f_q = E\left(\frac{A}{1-\mu} - \frac{B}{1+\mu}\right) = \text{Min. Principal Stress.}$$

If it is desired to construct a stress circle on the same centre as the strain circle then f_p and f_q must be in the form $A+B^1$ and $A-B^1$. Hence by rearranging the formula for f_p and f_q, this becomes :—

$$f_{pq} = \frac{E}{1-\mu}\left(A \pm \frac{1-\mu}{1+\mu}\cdot B\right) = \frac{E}{1-\mu}(A \pm B^1)$$

$$\text{where } B^1 = \frac{1-\mu}{1+\mu}\cdot B$$

This expression shows that a circle of radius $\frac{1-\mu}{1+\mu}\cdot B$ can be constructed about centre A, *i.e.*, concentric with the strain circle.

The value of the maximum principal stress will be :—

$f_p = \frac{E}{1-\mu}$ (Algebraically greater value of the strain at the intersection of the stress circle with the horizontal axis).

and $f_q = \frac{E}{1-\mu}$ (Algebraically smaller value of the strain at the intersection of the stress circle with the horizontal axis).

Example 1.

Suppose that three strain gauges are placed on a test surface in the form of a rectangular rosette, and that the measured strains are as follows :—

$e_1 = -5 \times 10^{-4}$ inches/inch.
$e_2 = 7 \times 10^{-4}$,, ,,
$e_3 = 6 \times 10^{-4}$,, ,,

Substituting these values in the formula for the principal strains, then :—

$$e_p = \frac{e_1+e_3}{2} + \sqrt{\left(\frac{e_1-e_3}{2}\right)^2 + \left(\frac{2e_2-e_1-e_3}{2}\right)^2} \cdot 10^{-4}$$

$$e_p = \frac{-5+6}{2} + \sqrt{\left(\frac{-5-6}{2}\right)^2 + \left(\frac{14+5-6}{2}\right)^2} \cdot 10^{-4}$$

$$e_p = \frac{1}{2} + \sqrt{\frac{121}{4} + \frac{169}{4}} \cdot 10^{-4}$$

$$e_p = \cdot 5 + \sqrt{72 \cdot 5} \cdot 10^{-4}$$

$$e_p = (\cdot 5 + 8 \cdot 51)\ 10^{-4}$$

$= + 9 \cdot 01 \times 10^{-4}$ inches/inch. Max. Principal Strain.

and $e_q = (\cdot 5 - 8 \cdot 51)\ 10^{-4}$

$e_q = -8 \cdot 01 \times 10^{-4}$ inches/inch. Min. Principal Strain.

$$\text{Tan } 2\theta = \frac{2e_2-e_1-e_3}{e_1-e_3}$$

$$\text{Tan } 2\theta = \frac{14+5-6}{-5-6} = -\frac{13}{11} = -1 \cdot 182$$

Whence $2\theta = 130°$ and $310°$

And $\theta = 65°$ and $155°$.

Hence max. principal strain has a value of $+9 \cdot 01 \times 10^{-4}$ inches/inch and acts in direction inclined at 65° to the direction of e_1.

The min. principal strain has a value of $-8 \cdot 01 \times 10^{-4}$ inches/inch and acts in a direction inclined at 155° to the direction of e_1 ·, *i.e.*, the maximum and minimum principal strains are mutually perpendicular to each other.

The corresponding principal stresses are derived from the formula

$$f_{pq} = E\left(\frac{A}{1-\mu} \pm \frac{B}{1+\mu}\right)$$

Taking values of Young's modulus E and Poisson's ratio μ as 13,500 tons/in.^{2} and ·3 respectively, then since

$$A = \frac{e_1+e_3}{2} = \frac{-5+6}{2} = \cdot 5 \times 10^{-4}$$

$$\text{and } B = \sqrt{\left(\frac{e_1-e_3}{2}\right)^2 + \left(\frac{2e_2-e_1-e_3}{2}\right)^2}$$

$$= \sqrt{\frac{121}{4} + \frac{169}{4}} = 8 \cdot 51 \times 10^{-4}$$

$$\text{then } f_{pq} = 1{\cdot}35 \times 10^4 \left(\frac{{\cdot}5}{1-{\cdot}3} \pm \frac{8{\cdot}51}{1+{\cdot}3} \right) \times 10^{-4}$$

$$f_{pq} = 1{\cdot}35\ ({\cdot}72 \pm 6{\cdot}55).$$

$f_p = +9{\cdot}81$ tons/in.^{2} = Maximum Principal Stress.

$f_q = -7{\cdot}84$ tons/in.^{2} = Minimum Principal Stress.

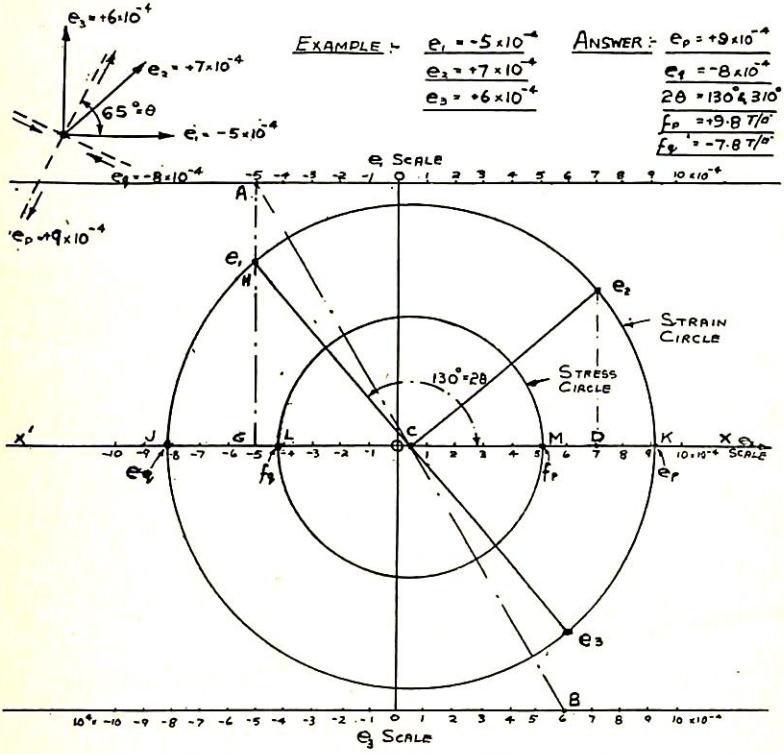


Fig. 16—Graphical Solution of Example 1.

Graphical Solution.

A quick graphical method of solving the rectangular strain rosette, due to Baumberger (Ref. 3), is demonstrated in Fig. 16. The construction is based on the Mohr strain circle.

The example just quoted and solved mathematically by means of the rectangular rosette formulae has been used to illustrate this graphical method of calculation.

The procedure to be adopted is as follows :—

(1) Plot e_1 at A on the upper scale.
(2) Plot e_3 at B on the lower scale.
(3) Join AB and intersection of this line with middle scale gives C.
(4) Plot e_2 at D on the middle scale.
(5) Transfer CD to GH on GA ordinate.
Plot CD = GH below G if D is to the left of C.
Plot CD = GH above G if D is to the right of C.
(6) Draw CH = radius of strain circle.
(7) Angle 2θ is given by angle HCX (θ is positive clockwise from e_1).
(8) Draw Strain Circle e_1, e_2, e_3 to find e_p = OK and e_q = OJ.
(9) Draw Stress Circle of radius $\left(\frac{1-\mu}{1+\mu}\right) \times$ CH.
(10) Find f_p and f_q from $\frac{E}{1-\mu} \times$ OM and $\frac{E}{1-\mu} \times$ OL.

The result for Example 1 shows that the element under consideration is in a state of almost pure shear; the condition for this being that the maximum and minimum principal stresses are equal in magnitude but opposite in sign. When this occurs the centre C of the stress and the strain circle is coincident with the origin—Fig. 16.

Example 2, Fig. 17, illustrates the method of solution when e_2 is negative. It will be seen that the circumference of the stress circle cuts the horizontal axis close to the origin, *i.e.*, the lateral stress f_q is small. This approaches the condition for simple tension.

Another condition for simple tension is that the lateral strain $= -\mu \times$ Longitudinal Strain. (μ = Poisson's ratio).

In Fig. 17 $\frac{QR}{RT} = \frac{B^1 - B}{B^1 + B}$

$$= \frac{\text{Stress Circle Rad} - \text{Strain Circle Rod}}{\text{Strain Circle Rad} + \text{Strain Circle Rod}}$$

Note.—(R assumed on origin).

$$,, = \frac{\left(\frac{1-\mu}{1+\mu} - 1\right) B}{\left(\frac{1-\mu}{1+\mu} + 1\right) B}$$

$$,, = \frac{1-\mu-1-\mu}{1-\mu+1+\mu} = \frac{-2\mu}{2}$$

$$\frac{QR}{RT} = -\mu.$$

i.e., Lateral Strain QR $= -\mu \times$ *Longitudinal Strain* RT.

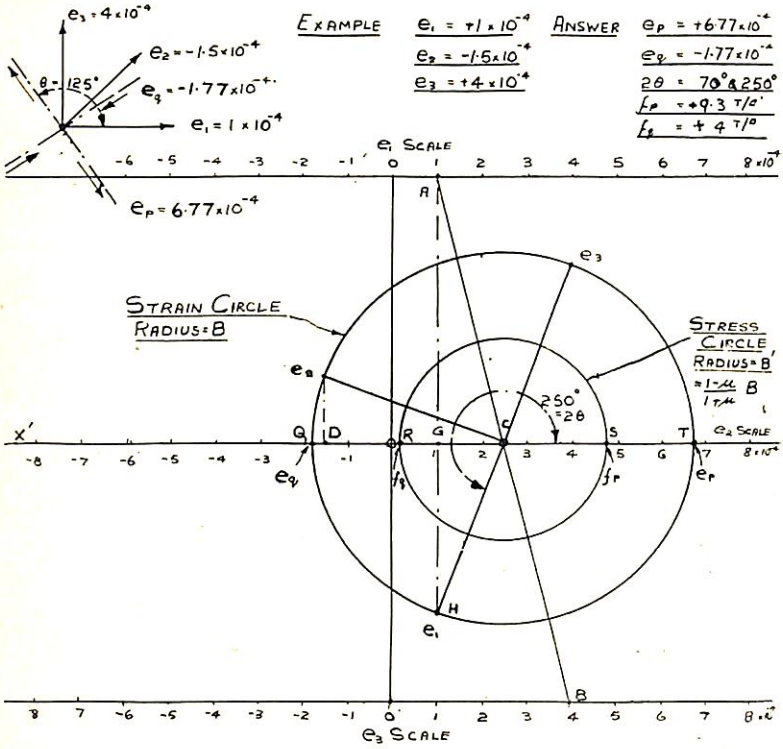


Fig. 17—Graphical Solution of Example 2.

Four-Gauge Rectangular or "Fan" Type Rosette.

A check on the accuracy of the observations can be made by making use of the fact that the sum of the strains observed in any two directions at right angles is a constant, *i.e.*, $(e_1 + e_3) = (e_2 + e_4)$ = sum of maximum and minimum principal strains $(e_p + e_q)$.

Hence if four strain measurements are made in directions spaced 45° apart so that $\theta_1 = 0°$, $\theta_2 = 45°$, $\theta_3 = 90°$, and $\theta_4 = 135°$, then from the solution of the equations e_1, e_2, e_3, e_4 respectively, the principal strains and their directions are given by :—

$$e_p = \frac{(e_1 + e_2 + e_3 + e_4)}{4} + \sqrt{\left(\frac{e_2 - e_4}{2}\right)^2 + \left(\frac{e_1 - e_3}{2}\right)^2}$$

$$e_q = \frac{(e_1+e_2+e_3+e_4)}{4} - \sqrt{\left(\frac{e_2-e_4}{2}\right)^2 + \left(\frac{e_1-e_3}{2}\right)^2}$$

and $\text{Tan } 2\theta = -\frac{e_2-e_4}{e_1-e_3}$

Equiangular Rosette.

Fig. 12c shows the arrangement known as the equiangular rosette. The requisite strain-gauge, specially designed for use with this particular method of strain measurement is shown in Fig. 12d.

If $\theta_1=0°$, $\theta_2=60°$, $\theta_3=120°$, the equations for the principal strains are :—

$$e_p = \frac{(e_1+e_2+e_3)}{3} + \sqrt{\left[e_1-\frac{1}{3}(e_1+e_2+e_3)\right]^2 + \left[\frac{1}{\sqrt{3}}(e_3-e_2)\right]^2}$$

$$e_q = \frac{(e_1+e_2+e_3)}{3} - \sqrt{\left[e_1-\frac{1}{3}(e_1+e_2+e_3)\right]^2 + \left[\frac{1}{\sqrt{3}}(e_3-e_2)\right]^2}$$

The directions of e_p and e_q can be found from the two values of :—

$$\text{Tan } 2\theta = \frac{\frac{1}{\sqrt{3}}(e_3-e_2)}{e_1-\frac{1}{3}(e_1+e_2+e_3)}$$

All the rosette formulae can be written in the form :—

$$e_p = A + B$$
$$\text{and } e_q = A - B$$

In each case the angle 2θ can be determined from the ratio of the two quantities inside the root sign of B.

A knowledge of these facts forms the basis for the design of various types of mechanical and electrical stress computers which have appeared from time to time in the technical press (Ref. 4).

Stress computers have been developed in fulfilment of the need for a rapid method of solving rosette equations.

A particularly neat and compact computer for rectangular rosettes, based on an original design by R. G. Manley, M.Sc. (London), is shown in Fig. 18. A detailed description of the prototype instrument appeared in "Engineering," Volume 164, No. 4263, pp. 340-1 (10-10-47).

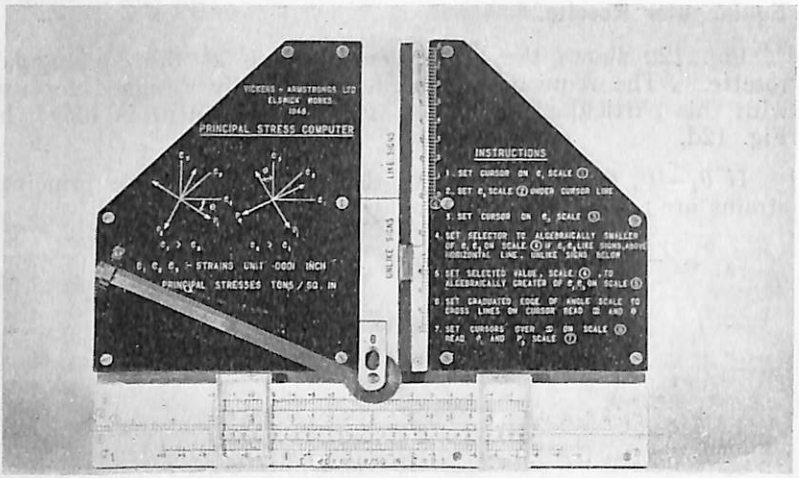

VICKERS - ARMSTRONGS LTD
ELSWICK WORKS
1946
PRINCIPAL STRESS COMPUTER
LIKE SIGNS
UNLIKE SIGNS
INSTRUCTIONS
1. SET CURSOR ON e_1 SCALE ①
2. SET e_2 SCALE ② UNDER CURSOR LINE
3. SET CURSOR ON e_3 SCALE ③
4. SET SELECTOR TO ALGEBRAICALLY SMALLER OF e_1 e_3 ON SCALE ④ IF e_1 e_3 LIKE SIGNS ABOVE HORIZONTAL LINE, UNLIKE SIGNS BELOW
5. SET SELECTED VALUE, SCALE ④, TO ALGEBRAICALLY GREATER OF e_1 e_3 ON SCALE ⑤
6. SET GRADUATED EDGE OF ANGLE SCALE TO CROSS LINES ON CURSOR READ ⑥ AND θ
7. SET CURSORS OVER ⑤ ON SCALE ⑤ READ P_1 AND P_2 SCALE ⑦
e_1 e_2 e_3 = STRAINS UNIT 000 INCH
PRINCIPAL STRESSES TONS / SQ. IN

Fig. 18—Rectangular Rosette Computer.

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